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Short Communication

Approximate analytical solutions for oscillation of a mass attached to a stretched elastic wire

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Abstract

Approximate analytical solutions for oscillation of a mass attached to a stretched elastic wire are presented. The approximation is based on combining Newton's method with the harmonic balance method. One iteration step can provide very accurate analytical approximate solutions for both small and large values of oscillation amplitude and parameter.

In dimensionless form, a mass attached to the center of a stretched elastic wire has the equation of motion [1]

$$\frac{d^2u}{dt^2}+u-\frac{\lambda u}{\sqrt{1+u^2}}=0,\quad u(0)=A,\quad \frac{du}{dt}(0)=0\qquad (0<\lambda<1),\tag{1}$$

which is an example of a conservative nonlinear oscillatory system having an irrational elastic item. The system will oscillate between symmetric bounds $[-A, A]$, and its period and the corresponding periodic solution are dependent on the amplitude A. Note that for small and large u, Eq. (1) becomes

$$\frac{d^2u}{dt^2}+(1-\lambda)u\approx 0,\ \ u\ll 1\quad \text{and}\quad \frac{d^2u}{dt^2}+u\approx 0,\ \ u\gg 1,\tag{2}$$

respectively. Consequently, for small A, $\omega\approx\sqrt{1-\lambda}$ while for large A, $\omega\approx 1$.

Rewriting Eq. (1) and applying the lowest order harmonic balance (HB) method [1,2], Mickens [1] obtained the analytical approximate period and corresponding periodic solution:

$$T_M(A)=2\pi/\omega_M(A),\quad u_M(t)=A\cos\omega_M(A)t,\quad \omega_M(A)=\sqrt{1-\lambda/\sqrt{1+A^2/2}}.\tag{3}$$

The HB method [1,2] is very difficult to construct higher-order analytical approximations because it requires analytical solutions of sets of complicated nonlinear algebraic equations. To improve the HB method, Wu and Li [3] presented an approach, which combines the HB method and linearization of nonlinear oscillation

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equation with respect to displacement increment only. This approach has been generalized and applied to other complex nonlinear oscillators [4–7]. For higher-order approximations, however, this approach results in a complex nonlinear algebraic equation in terms of unknown frequency and its analytical solution is again difficult. To overcome the problem, analytical approximations for oscillation of a mass attached to a stretched elastic wire based on a new approach is presented in this Letter. This new approach is obtained by combining Newton's method with the harmonic balance method. By using the approach, one obtains linear algebraic equations instead of nonlinear algebraic equations in each iteration. The complexity of the HB method is simplified accordingly.

Introducing a new independent variable $\tau = \omega t$, Eq. (1) can be rewritten as

$$\left(1+u^2\right)(\Omega u''+u)^2-\lambda^2u^2=0, \quad u(0)=A, \quad u'(0)=0, \tag{4}$$

where ($'$) denotes differentiation with respect to τ and $\Omega = \omega^2$. The new independent variable is chosen such that the solution of Eq. (4) is a periodic function of τ of period 2π. The corresponding period of the nonlinear oscillator is given by $T = 2\pi/\sqrt{\Omega}$. Since $f(u) = u - \lambda u/\sqrt{1+u^2}$ is an odd function of u, the periodic solution $u(\tau)$ can be represented by a Fourier series containing only odd multiples of τ, i.e.,

$$u(\tau)=\sum_{n=1}^{\infty} h_n \cos[(2n-1)\tau]. \tag{5}$$

Following the lowest order HB approximation, we set

$$u_1(\tau) = A\cos\tau. \tag{6}$$

which satisfies the initial conditions in Eq. (4). Substituting Eq. (6) into Eq. (4), expanding the resulting expression in a trigonometric series and setting the constant term to zero yield

$$A^2\left(4-4\lambda^2-8\Omega_1+4\Omega_1^2\right)+A^4\left(3-6\Omega_1+3\Omega_1^2\right)=0, \tag{7}$$

which can be solved for Ω_1 as a function of A. There are two solutions for Ω_1, i.e.,

$$\Omega_1(A) = 1 \pm \lambda/\sqrt{1+3A^2/4}. \tag{8}$$

Since an increase in the amplitude A results in increasing angular frequency as discussed earlier about Eq. (2), the negative sign in Eq. (8) should be selected, i.e.,

$$\Omega_1(A) = 1 - \lambda/\sqrt{1+3A^2/4}. \tag{9}$$

Therefore, the first analytical approximate period and corresponding periodic solution are

$$T_1(A) = 2\pi/\sqrt{\Omega_1(A)}, \quad u_1(t) = A\cos\tau, \quad \tau = \sqrt{\Omega_1(A)t}. \tag{10}$$

Note that the solution in Eq. (10) is different from that in Eq. (3) obtained by Mickens [1].

Using $u_1(\tau)$ and $\Omega_1(A)$ as initial approximations to the solution of Eq. (4), we combine Newton's method and the HB method to solve Eq. (4). The first step is the Newton procedure. The periodic solution and the square of frequency of Eq. (4) can be expressed as

$$u = u_1 + \Delta u_1, \quad \Omega = \Omega_1 + \Delta\Omega_1. \tag{11}$$

Substituting Eq. (11) into Eq. (4) and linearizing with respect to the correction terms Δu_1 and $\Delta\Omega_1$ lead to

$$\begin{aligned}&\left(1+u_1^2+2u_1\Delta u_1\right)\left(\Omega_1^2u_1''^2+u_1^2+2\Omega_1u_1u_1''\right)+2\left(1+u_1^2\right)\left(\Omega_1^2u_1''\Delta u_1''+\Omega_1u_1''^2\Delta\Omega_1+u_1\Delta u_1\right.\\&\quad\left.+\Omega_1u_1\Delta u_1''+\Omega_1u_1''\Delta u_1+u_1u_1''\Delta\Omega_1\right)-\lambda^2\left(u_1^2+2u_1\Delta u_1\right)=0,\\&\quad\Delta u_1(0)=0,\ \ \Delta u_1'(0)=0,\end{aligned} \tag{12}$$

where Δu_1 is a periodic function of τ of period 2π, and both Δu_1 and $\Delta\Omega_1$ are to be determined.

The resulting linear equation in Δu_1 and $\Delta\Omega_1$ in Eq. (12) will be solved by the HB method. The second approximate solution to Eq. (12) can be developed by setting $\Delta u_1(\tau)$ as

$$\Delta u_1(\tau) = x_1(\cos\tau - \cos 3\tau), \tag{13}$$

which satisfies the initial condition in Eq. (12) at the outset. Substituting Eqs. (10) and (13) into Eq. (12), expanding the resulting expression in a trigonometric series and setting the constant term and the coefficients of $\cos 2\pi$ to zero, respectively, yield

$$4\left(1-\lambda^2-2\Omega_1+\Omega_1^2\right)A^2+\left(3-6\Omega_1+3\Omega_1^2\right)A^4-2\left(4A^2+3A^4-4A^2\Omega_1-3A^4\Omega_1\right)\Delta\Omega_1 + 8\left(A+A^3-A\lambda^2-2A\Omega_1+A\Omega_1^2-A^3\Omega_1^2\right)x_1=0, \tag{14}$$

$$\left(1-\lambda^2-2\Omega_1+\Omega_1^2\right)A^2+\left(1-2\Omega_1+\Omega_1^2\right)A^4-2\left(A^2+A^4-A^2\Omega_1-A^4\Omega_1\right)\Delta\Omega_1 + \left(A^3+16A\Omega_1-16A\Omega_1^2+10A^3\Omega_1-11A^3\Omega_1^2\right)x_1=0. \tag{15}$$

Solving Eqs. (14) and (15) for x_1 and $\Delta\Omega_1$, and applying Eq. (9) obtain

$$x_1=-\frac{A^3\left(4+3A^2\right)\lambda}{4F(A)},\quad \Delta\Omega_1=\frac{A^4\left(7\lambda^2-4\lambda\sqrt{4+3A^2}\right)}{2\sqrt{4+3A^2}F(A)}, \tag{16}$$

where

$$F(A)=2\left(16+20A^2+5A^4\right)\sqrt{4+3A^2}-\left(64+78A^2+19A^4\right)\lambda.$$

Therefore, the second analytical approximations to period and periodic solution of the nonlinear oscillator is

$$T_2(A)=2\pi/\sqrt{\Omega_2(A)},\quad u_2(\tau)=u_1(\tau)+\Delta u_1(\tau)=X(A)\cos\tau+Y(A)\cos 3\tau,\quad \tau=\sqrt{\Omega_2(A)}t, \tag{17}$$

where

$$\Omega_2(A)=1-\frac{2\lambda}{\sqrt{4+3A^2}}+\frac{A^4\left(7\lambda^2-4\lambda\sqrt{4+3A^2}\right)}{2\sqrt{4+3A^2}F(A)},\quad X=A-\frac{A^3\left(4+3A^2\right)\lambda}{4F(A)},\quad Y=\frac{A^3\left(4+3A^2\right)\lambda}{4F(A)}.$$

It should be clear how the procedure works for constructing further higher-order approximate solutions. Higher-order approximate solutions are more accurate but somewhat complicated, hence they are omitted here.

Table 1
Comparison of the approximate periods with exact period

A	T_e	T_M/T_e	T_1/T_e	T_2/T_e	T_e	T_M/T_e	T_1/T_e	T_2/T_e
(I) $\lambda = 0.1$					(II) $\lambda = 0.5$			
0.1	6.621688	1.000069	1.000000	1.000000	8.869257	1.000616	0.999999	1.000000
0.4	6.603056	1.000932	0.999974	1.000009	8.653029	1.008046	0.999729	1.000057
0.7	6.571430	1.002096	0.999842	1.000052	8.316817	1.016943	0.998530	1.000333
1	6.537508	1.002913	0.999622	1.000126	7.992133	1.021992	0.996803	1.000757
4	6.378382	1.001915	0.999027	1.000160	6.809366	1.010797	0.994236	1.000720
7	6.339461	1.001085	0.999288	1.000054	6.581398	1.005780	0.996078	1.000206
10	6.322939	1.000744	0.999462	1.000020	6.490208	1.003883	0.997119	1.000061
(III) $\lambda = 0.75$					(IV) $\lambda = 0.95$			
0.1	12.49674	1.001836	0.999994	1.000000	27.15679	1.011058	0.999889	0.999976
0.4	11.65250	1.022103	0.999029	1.000069	19.78073	1.084019	0.992948	0.998818
0.7	10.53734	1.041411	0.995671	1.000452	14.73436	1.106253	0.984488	0.998716
1	9.625405	1.048464	0.991951	1.001074	12.07528	1.098603	0.980077	0.999840
4	7.129819	1.017585	0.990244	1.000930	7.423433	1.023903	0.986242	1.000933
7	6.748172	1.009037	0.993718	1.000229	6.891635	1.011845	0.991589	1.000189
10	6.602092	1.005986	0.995480	1.000047	6.696117	1.007753	0.994057	1.000007

For the nonlinear oscillator, the exact period is

$$T_e(A) = 4\int_0^{\pi/2} \left[1 - 2\lambda/\left(\sqrt{1 + A^2\sin^2\theta} + \sqrt{1 + A^2}\right)\right]^{-1/2} \mathrm{d}\theta, \tag{18}$$

with details of derivation presented in Appendix A.

Table 1 shows the ratios of the approximate periods T_{M}, T_1, T_2 in Eqs. (3), (10) and (17), respectively, to the exact period T_e in Eq. (18). Furthermore, we have

$$\lim_{A\to 0^+} \frac{T_M}{T_e} = \lim_{A\to +\infty} \frac{T_M}{T_e} = \lim_{A\to 0^+} \frac{T_1}{T_e} = \lim_{A\to +\infty} \frac{T_1}{T_e} = \lim_{A\to 0^+} \frac{T_2}{T_e} = \lim_{A\to +\infty} \frac{T_2}{T_e} = 1. \tag{19}$$

From Table 1 and Eq. (19), it can be observed that Eq. (17) yield excellent analytical approximate periods for both small and large values of oscillation amplitude A and parameter λ. Furthermore, it has shown that the first analytical approximate period in Eq. (10) is more accurate than that in Eq. (3) obtained by Mickens [1]. For $A = 1$, $\lambda = 0.75$, 0.95 and $A = 4$, $\lambda = 0.1$, 0.5 the (numerical) exact periodic solution $u_e(t)$ obtained by numerically integrating Eq. (1) and the approximate periodic solutions $u_M(t)$, $u_1(t)$ and $u_2(t)$ computed by Eqs. (3), (10) and (17), respectively, are plotted in Figs. 1–4. These figures show that the proposed solutions in

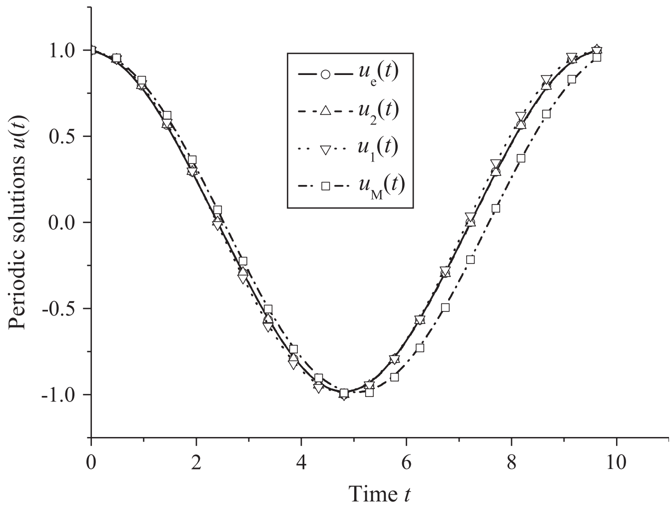

Fig. 1. Comparison of approximate and exact periodic solutions for $A = 1$, $\lambda = 0.75$.

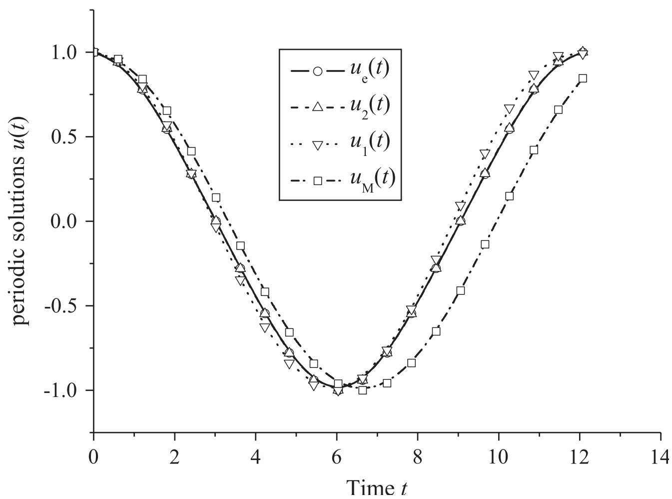

Fig. 2. Comparison of approximate and exact periodic solutions for $A = 1$, $\lambda = 0.95$.

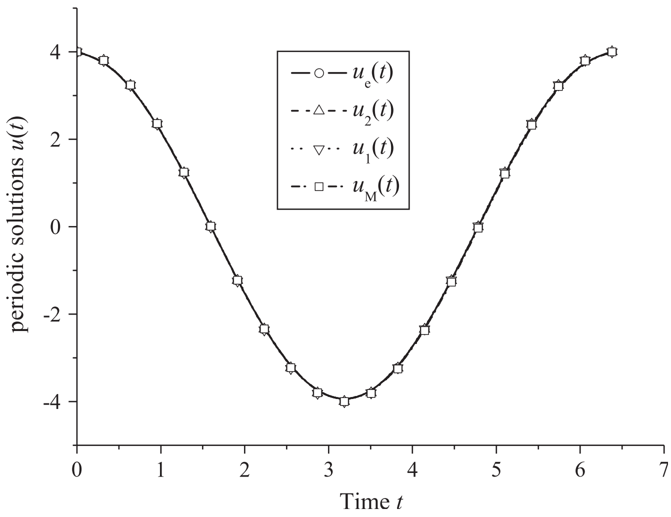

Fig. 3. Comparison of approximate and exact periodic solutions for $A = 4$, $\lambda = 0.1$.

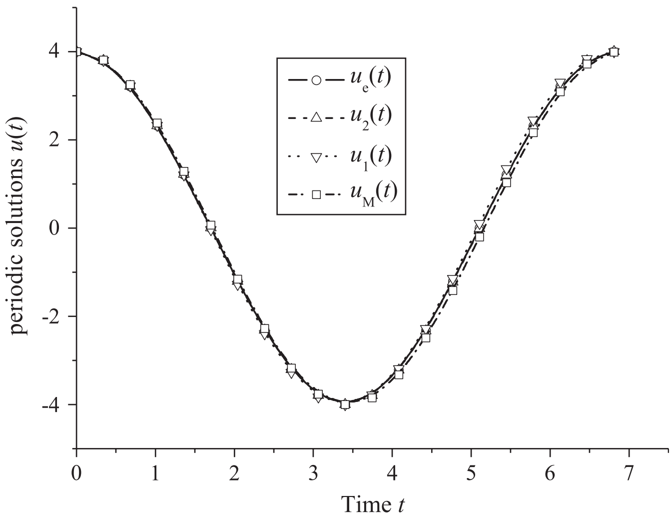

Fig. 4. Comparison of approximate and exact periodic solutions for $A = 4$, $\lambda = 0.5$.

Eqs. (10) and (17) are in excellent agreement with exact periodic solutions for both small and large values of oscillation amplitude A and parameter λ.

In summary, a new approach has been presented for analytically solving the oscillator describing a mass attached to a stretched elastic wire. The iteration yields rapid convergence with respect to exact solution. The result is valid for both small and large values of oscillation amplitude and parameter, including the limiting cases of oscillation amplitude approaching zero and infinity.

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Appendix A. Derivation of the exact period

The exact period in Eq. (18) is derived here. Integrating the differential equation and using the corresponding initial conditions in Eq. (1) yields the first integral as follows:

$$\left(\frac{\mathrm{d}u}{\mathrm{d}t}\right)^2 + u^2 - 2\lambda\sqrt{1+u^2} = A^2 - 2\lambda\sqrt{1+A^2}. \tag{A.1}$$

Solving Eq. (A.1) for dt gives

$$\mathrm{d}t = \pm\frac{\mathrm{d}u}{\sqrt{(A^2-u^2)\left[1-2\lambda/\left(\sqrt{1+u^2}+\sqrt{1+A^2}\right)\right]}}. \tag{A.2}$$

The exact period of the oscillation is four times the time taken by the mass to move from $u = 0$ to $u = A$. Therefore,

$$T_e(A) = 4\int_0^A \frac{\mathrm{d}u}{\sqrt{(A^2-u^2)\left[1-2\lambda/\left(\sqrt{1+u^2}+\sqrt{1+A^2}\right)\right]}}. \tag{A.3}$$

A new variable θ is defined as

$$u = A\sin\theta. \tag{A.4}$$

Introducing Eq. (A.4) into Eq. (A.3) leads to

$$T_e(A) = 4\int_0^{\pi/2}\left[1-2\lambda/\left(\sqrt{1+A^2\sin^2\theta}+\sqrt{1+A^2}\right)\right]^{-1/2}\mathrm{d}\theta. \tag{A.5}$$

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